Schurman MINT Model - Valuing A Minority Interest
Part II - Incorporating A Stochastic Payout Ratio

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September, 2012

In Part I we developed the Base Case Model where the payout ratio Φ was a non-random variable such that the sole component of minority shareholder risk was the risk associated with company control cash flows. If control cash flows was the only random variable then the product of random control cash flows and a non-random payout ratio would not impact the minority shareholder's risk profile. Once we make the payout ratio a random variable then we have two sources of risk - (1) the volatility of control cash flows and (2) the volatility of the payout ratio. In Part II we will develop the mathematics that revises the Base Case Model of Part I to incorporate a stochastic payout ratio that may increase or decrease over time.

In Part I we presented a hypothetical problem pertaining to the valuation of a minority interest. The table below presents the valuation assumptions from Part I and two new assumptions applicable to Part II...

Table 1 - Model Parameters

Variable	Description	Value	Reference
k	Continous time discount rate	0.1823	From Part I
g	Continous time growth rate	0.0392	From Part I
y	Continous time dividend yield	0.1431	From Part I
θ	Minority ownership percentage	0.2000	From Part I
Φ	Payout ratio	0.5000	From Part I
λ	Company sale hazard rate	0.1000	From Part I
σ_m	Volatility of company returns	0.5400	From Part I
$\rho_{m,mrkt}$	Correlation company and market returns	0.6000	From Part I
σ_n	Volatility of the payout ratio	0.1500	New assumption
$\rho_{n,mrkt}$	Correlation payout and market returns	0.5000	New assumption
σ_{mrkt}	Volatility of market returns	0.1800	From Part I
rfr	Annual risk-free rate	0.0560	From Part I

Modeling Company Control Value As A Stochastic Variable

In Part I we determined that the control value at time $t=0$ of our hypothetical company was...

$$V_0 = \int\limits_0^\infty C_0\, e^{gt} e^{-kt}\,\delta t = \frac{C_0}{k-g} = \frac{1{,}000{,}000}{0.1823 - 0.0392} = 6{,}988{,}000 \tag{1}$$

In Equation (1) above control free cash flow at time zero (C_0) is \$1,000,000, the continuous time risk-adjusted discount rate (k) is 18.23%, the continous time free cash flow growth rate (g) is 3.92% and t is time in years.

In Part I we also determined that the equation for expected company control value at any future time t was...

$$\mathbb{E}\Big[V_t\Big] = V_0\, e^{gt} \tag{2}$$

If we take the derivative of Equation (2) above with respect to time then the expected change in company control value over time is...

$$\delta\,\mathbb{E}\Big[V_t\Big] = g\,V_0\,e^{gt}\,\delta t = g\,\mathbb{E}\Big[V_t\Big]\,\delta t \tag{3}$$

We will model stochastic company control value at time t to be a function of a deterministic drift and an innovation term, which is the source of risk for this variable. Using Equation (3) as our guide we will model the change in company control value ($\delta\, V_t$) via the following stochastic differential equation (SDE)...

$$\delta\, V_t = g\, V_t\, \delta t + \sigma_m\, V_t\, \delta M_t \;\; \text{...where...} \;\; \delta\, M_t \sim N[0, \delta\, t] \tag{4}$$

In the SDE above σ_m is annual return volatility and $\delta\, M_t$ is the change in the driving Brownian motion over the infinitesimally small time interval $[t, t+\delta t]$. The solution to the SDE, which is the equation for company control value at time t conditioned on the value of the Brownian motion M at time t, is...

$$V_t = V_0\, Exp\left\{ \left(g - \frac{1}{2}\sigma_m^2 \right) t + \sigma_m\, M_t \right\} \;\; \text{...where...} \;\; M_t \sim N[0, t] \tag{5}$$

After normalizing Equation (5) the equation for the stochastic company control value at time t becomes...

$$V_t = V_0\, Exp\left\{ \left(g - \frac{1}{2}\sigma_m^2 \right) t + \sigma_m\, \sqrt{t}\, X_t \right\} \;\; \text{...where...} \;\; X_t \sim N[0, 1] \tag{6}$$

Modeling The Payout Ratio As A Stochastic Variable

We will model the stochastic payout ratio at time t to be a function of a deterministic drift, which in our case is zero, and an innovation term, which is the source of risk for this variable. As we did in Equation (4) above we will model the change in the payout ratio ($\delta\, \Phi_t$) via the following stochastic differential equation (SDE)...

$$\delta\, \Phi_t = \sigma_n\, \Phi_t\, \delta N_t \;\; \text{...where...} \;\; \delta\, N_t \sim N[0, \delta\, t] \tag{7}$$

In the SDE above σ_n is annual volatility and $\delta\, N_t$ is the change in the driving Brownian motion over the infinitesimally small time interval $[t, t+\delta t]$. The solution to the SDE, which is the equation for the payout ratio at time t conditioned on the value of the Brownian motion N at time t, is...

$$\Phi_t = \Phi_0\, Exp\left\{ - \frac{1}{2}\sigma_n^2\, t + \sigma_n\, N_t \right\} \;\; \text{...where...} \;\; N_t \sim N[0, t] \tag{8}$$

After normalizing Equation (8) the equation for the stochastic payout ratio at time t becomes...

$$\Phi_t = \Phi_0\, Exp\left\{ - \frac{1}{2}\sigma_n^2\, t + \sigma_n\, \sqrt{t}\, Y_t \right\} \;\; \text{...where...} \;\; Y_t \sim N[0, 1] \tag{9}$$

The Revised Model That Incorporates A Stochastic Payout Ratio

In Part I we developed the Base Case Model where the value of a minority shareholder's ownership interest is the discounted value of the minority shareholder's expected cash flow over the time interval $[0, \infty]$ given that the payout ratio phi (Φ) is a non-random variable. The valuation equation from Part I was...

$$\text{Value of Minority Shares} = \int\limits_0^\infty y\, \Phi\, \theta\, V_0\, e^{(g-k-\lambda)t}\, \delta t + \int\limits_0^\infty \lambda\, \theta\, V_0\, e^{(g-k-\lambda)t}\, \delta t \tag{10}$$

The solution to Equation (10) above, which is the Base Case Model from Part I, is...

$$\text{Value of Minority Shares} = \frac{y\, \Phi\, \theta\, V_0}{k + \lambda - g} + \frac{\lambda\, \theta\, V_0}{k + \lambda - g} \tag{11}$$

The first half of Equations (10) and (11) is the present value of the expected dividend stream and the second half is the present value of the expected liquidating distribution. Because the value of the liquidating distribution does not contain the payout ratio phi we do not have to rewrite that part of the equation when revising the Base Case Model to incorporate a stochastic payout ratio. We can rewrite the first half of Equation (10) as...

$$\text{Value of Minority Dividends} = \int\limits_0^\infty y\, \Phi\, \theta\, V_0\, e^{(g-k-\lambda)t}\, \delta t = \int\limits_0^\infty y\, \theta\, \Phi_t\, V_t\, e^{-\lambda t}\, e^{-kt}\, \delta t \tag{12}$$

Note that in Equation (12) above V_t is the now-stochastic company control value, Φ_t is the now-stochastic payout ratio, $\exp(-\lambda t)$ is the probability of receiving the dividend (i.e. the company is not sold prior to time t) and $\exp(-kt)$ is the discount factor.

If we take Equation (12) and remove the integral notation, remove the probability of receiving the dividend, remove the discount factor, substitute V_t with Equation (6) and substitute Φ_t with Equation (9) then the equation for the stochastic dividend received by the minority shareholder over the infinitesimally small time interval $[t, t+\delta t]$ where D_t is the annualized dividend at time t is...

$$\begin{aligned} D_t\,\delta t &= y\,\theta\,\Phi_0\,Exp\left\{-\frac{1}{2}\sigma_n^2\,t+\sigma_n\,\sqrt{t}\,Xt\right\}V_0\,Exp\left\{\left(g-\frac{1}{2}\sigma_m^2\right)t+\sigma_m\,\sqrt{t}\,Y_t\right\}\delta t \\ &= y\,\theta\,\Phi_0\,V_0\,Exp\left\{\left(g-\frac{1}{2}\sigma_m^2-\frac{1}{2}\sigma_n^2\right)t+\sigma_m\,\sqrt{t}\,X_t+\sigma_n\,\sqrt{t}\,Y_t\right\}\delta t \end{aligned} \tag{13}$$

The effect of stochastic dividend Equation (13) above on minority share valuation is twofold...

1) The dividend equation now has two sources of risk (X_t and Y_t) that may or may not be correlated. Depending on the sign and magnitude of the correlation coefficient the discount rate used in the valuation of the dividend stream will increase or decrease. This will affect the denominator of the valuation equation.
2) Given two lognormally-distributed random variables (company control value and the payout ratio) the expectation of the product of the two random variables is not the product of the expectations. Depending on the correlation coefficient the cash flows used in the valuation of the dividend stream will increase or decrease. This will affect the numerator of the valuation equation.

The Numerator Effect

Given that the multipliers A and B are constants and the random variates x and y are normally-distributed the expected value of the product of two lognormally-distributed random variates is...

$$\mathbb{E}\left[A\,e^x\,B\,e^y\right]=A\,B\,e^{\mu_x+\mu_y+\frac{1}{2}(\sigma_x^2+\sigma_y^2+2\rho_{x,y}\sigma_x\sigma_y)} \tag{14}$$

Note: Reference the white paper 'The Mean And Variance Of The Product Of Two Lognormally-Distributed Random Variates (Schurman - September, 2012)' for a full derivation of the equation above.

Using Equations (13) above we want to find the expectation of...

$$\mathbb{E}\left[\Phi_0\,Exp\left\{-\frac{1}{2}\sigma_n^2\,t+\sigma_n\,\sqrt{t}\,Xt\right\}V_0\,Exp\left\{\left(g-\frac{1}{2}\sigma_m^2\right)t+\sigma_m\,\sqrt{t}\,Y_t\right\}\right] \tag{15}$$

We can solve the expectation in Equation (15) by mapping that equation to Equation (14) above. If we make the following mappings...

$$A=V_0\ ,\ B=\Phi_0\ ,\ \mu_x=\left(g-\frac{1}{2}\sigma_m^2\right)t\ ,\ \mu_y=-\frac{1}{2}\sigma_n^2\,t\ ,\ \sigma_x=\sigma_m\,\sqrt{t}\,Y_t\ ,\ \sigma_y=\sigma_n\,\sqrt{t}\,Xt \tag{16}$$

...then the expected value of the product of the stochastic control value at time t and the stochastic penalty rate at time t is...

$$\mathbb{E}\left[\Phi_t\,V_t\right]=\Phi_0\,V_0\,Exp\left\{\left(g+\rho_{m,n}\,\sigma_m\,\sigma_n\right)t\right\} \tag{17}$$

Per Equation (17) we need the correlation of company control value and the payout ratio, which is not given to us in Table 1 above. In Table 1 the correlation of company returns and market returns is $\rho_{m,mrkt}=0.60$ and the correlation of the payout ratio and market returns is $\rho_{n,mrkt}=0.50$. For the Denominator Effect below we modeled both company returns and the payout ratio to be a function of market returns via the Gaussian Copula. It can be shown that when the common factor for both company returns and the payout ratio is market returns (and the residuals are uncorrelated) then the correlation betwen company returns and the payout ratio is...

$$\rho_{m,n}=\rho_{m,mrkt}\times\rho_{n,mrkt} \tag{18}$$

Using Equation (18) we can rewrite Equation (14) as...

$$\mathbb{E}\bigg[\Phi_t\,V_t\bigg] = \Phi_0\,V_0\,Exp\bigg\{\bigg(g + \rho_{m,mrkt}\,\rho_{n,mrkt}\,\sigma_m\,\sigma_n\bigg)\,t\bigg\} \tag{19}$$

The revised growth rate g-hat ($\hat{g}$) that we will use the value the dividend stream is therefore...

$$\begin{aligned}\hat{g} &= g + \rho_{m,mrkt}\,\rho_{n,mrkt}\,\sigma_m\,\sigma_n\\ &= 0.0392 + (0.60)(0.50)(0.54)(0.15)\\ &= 0.0635\end{aligned} \tag{20}$$

Note that when correlation is positive the expectation of the stochastic company control value and the stochastic payout ratio is greater than the product of their expectations such that expected cash flows increase.

The Denominator Effect

Per the white paper 'Derivation Of The Capital Asset Pricing Model - Two Sources Of Uncertainty (Schurman - August, 2012)' the CAPM beta applicable to a dividend stream with two sources of uncertainty is...

$$\beta = \frac{\sigma_m\,\rho_{m,mrkt} + \sigma_n\,\rho_{n,mrkt}}{\sigma_{mrkt}} \tag{21}$$

...where $\rho_{m,mrkt}$ is the correlation of control value returns with market returns and $\rho_{n,mrkt}$ is the correlation of the payout ratio with market returns.

Per Equation (21) and the assumptions in Table 1 the new CAPM beta applicable to our dividend stream is...

$$\beta = \frac{\sigma_m\,\rho_{m,mrkt} + \sigma_n\,\rho_{n,mrkt}}{\sigma_{mrkt}} = \frac{(0.54)(0.60) + (.15)(0.50)}{0.18} = 2.22 \tag{22}$$

Using the CAPM beta from Equation (22) and the assumptions in Table 1 the revised CAPM annual discrete-time discount rate applicable to our dividend stream is...

$$\text{CAPM discount rate} = 0.056 + 2.22 \times 0.08 = 0.2333 \tag{23}$$

The revised discount rate k-hat ($\hat{k}$) that we will use the value the dividend stream is therefore...

$$\hat{k} = \text{Continuous-time discount rate} = \ln(1 + 0.2333) = 0.2097 \tag{24}$$

Note that when the correlation of company returns with market returns is positive and the correlation of the payout ratio with market returns is positive then risk increases such that the discount rate increases.

The Revised Model Value Of The Minority Shares

Using the Base Case Model as defined by Equation (11) above the value of the minority shares per Part I was...

$$\begin{aligned}\text{Value of Minority Shares} &= \frac{y\,\Phi\,\theta\,V_0}{k+\lambda-g} + \frac{\lambda\,\theta\,V_0}{k+\lambda-g}\\ &= \frac{(0.1431)(0.5000)(0.2000)(6{,}988{,}000)}{0.1823 + 0.1000 - 0.0392} + \frac{(0.1000)(0.2000)(6{,}988{,}000)}{0.1823 + 0.1000 - 0.0392}\\ &= 411{,}000 + 575{,}000\\ &= 986{,}000\end{aligned} \tag{25}$$

The value determined via Equation (25) above assumes that the payout ratio is non-random. When we change that assumption then the new equation for minority share value when the revised growth rate per Equation (20) and the revised cost of capital per Equation (24) is substituted into the first half of Equation (11) above the value of the minority shares per the revised model is...

$$\begin{aligned}
\text{Value of Minority Shares} &= \frac{y\,\Phi\,\theta\,V_0}{\hat{k}+\lambda-\hat{g}} + \frac{\lambda\,\theta\,V_0}{k+\lambda-g} \\
&= \frac{(0.1431)(0.5000)(0.2000)(6{,}988{,}000)}{0.2097+0.1000-0.0635} + \frac{(0.1000)(0.2000)(6{,}988{,}000)}{0.1823+0.1000-0.0392} \\
&= 415{,}000 + 575{,}000 \\
&= 990{,}000
\end{aligned} \tag{26}$$

Note that for all intents and purposes the value of the minority shares did not change from Part I to Part II because the numerator and denominator effects offset each other. Note that this may not always be the case.